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Collapse of unit horizontal bundles equipped with a metric of Cheeger–Gromoll type

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ABSTRACT

We study unit horizontal bundles associated with Riemannian submersions. First we investigate metric properties of an arbitrary unit horizontal bundle equipped with a Riemannian metric of the Cheeger–Gromoll type. Next we examine it from the Gromov–Hausdorff convergence theory point of view, and we state a collapse theorem for unit horizontal bundles associated with a sequence of warped Riemannian submersions.

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1. Introduction and preliminaries

1.1. Introduction

Recently in [2], M. Benyounes, E. Loubeau and C.M. Wood introduced a new class of natural metrics of Cheeger–Gromoll type on the vector bundle over a Riemannian manifold. These metrics, $\mathbf{h}_{p,q}$, $p, q \in \mathbb{R}$, $q \geq 0$, called (p, q) -metrics, generalize Sasaki metric [7] and Cheeger–Gromoll metric [6] on TM .

Although (p, q) -metrics have been discovered together with some new harmonics maps, the geometry of (p, q) -geometry of the tangent bundle is of the independent interest [3].

In the present paper we combine a technique of (p, q) -metrics and Riemannian submersions with the Gromov–Hausdorff distance theory (GH-distance theory).

First, we investigate the unit horizontal bundle \tilde{E}^1 , associated with a Riemannian submersion $P : \tilde{M} \rightarrow M$. The total space of \tilde{E}^1 consists of the all unit vectors in $T\tilde{M}$ which are orthogonal to the fibres of P . The differential $\tilde{P} = P_*$ maps \tilde{E}^1 into SM —the unit sphere bundle over M . Let us equip \tilde{E}^1 and SM with the (p, q) -metric $\tilde{\mathbf{h}}$ and \mathbf{h} . We ask when $\tilde{P} : (\tilde{E}^1, \tilde{\mathbf{h}}) \rightarrow (SM, \mathbf{h})$ is a Riemannian submersion. We prove (Proposition 2.1) that $\tilde{P} : \tilde{E}^1 \rightarrow SM$ is a Riemannian submersion iff the horizontal distribution of P is integrable. This assertion seems to be of independent interest.

Next, we combine the result from Proposition 2.1 with Theorem 2 from [12], and we obtain Collapse Theorem (Theorem 2.2) for the unit horizontal bundle. We prove that the sequence of unit horizontal bundles $(E_n^1)_{n \in \mathbb{N}}$ associated with the

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sequence of warped Riemannian submersions $(P: \tilde{M}_n \rightarrow M)_{n \in \mathbb{N}}$ converges (in GH-topology) to the unit sphere bundle SM iff $(\tilde{M}_n)_{n \in \mathbb{N}}$ converges to M . An application of this theorem is given in Section 2.3

In fact, Theorem 2.2 asserts that this natural construction is continuous in the GH-topology. Some other examples showing the continuity of a natural constructions can be found in [8] by H. Li, where the author proves continuity of θ -deformations, and in the recent paper of P.G. Walczak [11] where the continuity of spaces of probability measures associated with a Riemannian manifolds is studied.

1.2. Riemannian submersions

We briefly review basic facts of Riemannian submersions. For more details we refer to [4, Ch. 9].

In this paper the term *submersion* shall always refer to a smooth surjective map $P: \tilde{M} \rightarrow M$ with $\dim M \leq \dim \tilde{M}$ of maximal rank. Recall that $P: (\tilde{M}, \tilde{g}) \rightarrow (M, g)$ is a *Riemannian submersion* if its tangent bundle $T\tilde{M}$ splits as a direct orthogonal sum $T\tilde{M} = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \ker P_*$ is the *vertical subbundle* and $\mathcal{H} = \mathcal{V}^\perp$ is the *horizontal subbundle*, and the following condition holds: for every $U, V \in \mathcal{H}_{\tilde{x}}$, $g(P_*U, P_*V) = \tilde{g}(U, V)$. If $W \in T\tilde{M}$ then $W = \mathcal{H}W + \mathcal{V}W$ denotes the corresponding orthogonal splitting. To indicate a submersion we work with, we often write \mathcal{H}^P and \mathcal{V}^P instead of \mathcal{H} and \mathcal{V} .

A vector field \tilde{X} (resp. \tilde{U}) is horizontal (resp. vertical) if $\tilde{X} \in \Gamma(\tilde{M}, \mathcal{H}^P)$ (resp. $\tilde{U} \in \Gamma(\tilde{M}, \mathcal{V}^P)$). The vector field \tilde{X} is called *basic* if \tilde{X} is horizontal and there exists a vector field X on M such that $P_*\tilde{X} = X$. There is one-to-one correspondence $X \rightarrow \tilde{X}$ between vector fields on M and basic vector fields on \tilde{M} . If X is a vector field on M then the corresponding basic vector field \tilde{X} is called the *horizontal lift* of X . If the vector fields \tilde{X}, \tilde{Y} are basic and the vector field \tilde{U} is vertical then $\tilde{g}(\tilde{X}, \tilde{Y})$ is constant along the fibres of P , and $[\tilde{X}, \tilde{U}]$ is also vertical. Moreover, the horizontal part of the Lie bracket $[\tilde{X}, \tilde{Y}]$ coincide with the horizontal lift $\widetilde{[X, Y]}$.

Let $\tilde{\nabla}$ and ∇ be the Levi-Civita connections of \tilde{g} and g , respectively. The connection \tilde{D} in \mathcal{H}^P is simply the horizontal projection $\mathcal{H}\tilde{\nabla}$. Since $\tilde{\nabla}$ is Riemannian, so \tilde{D} is.

Lemma 1.1. Suppose that \tilde{X}, \tilde{Y} are basic and \tilde{U} is a vertical vector field. Put $X = P_*\tilde{X}$ and $Y = P_*\tilde{Y}$. Then

- (i) $P_*\tilde{D}_{\tilde{X}}\tilde{Y} = \nabla_X Y$, and $\tilde{D}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla_X Y}$,
- (ii) $\tilde{g}(\tilde{D}_{\tilde{U}}\tilde{X}, \tilde{Y}) = -\frac{1}{2}\tilde{g}(\tilde{U}, [\tilde{X}, \tilde{Y}])$.

Lemma 1.2. Let us suppose that \tilde{M} is compact.

- (i) Let $\{x_1, \dots, x_k\}$ be an ε -net in M , and let for every $i = 1, \dots, k$, $\{\tilde{x}_{i1}, \dots, \tilde{x}_{il(i)}\}$ be an ε -net in the fibre $\tilde{M}_{x_i} = P^{-1}(x_i)$. Then the set $\{\tilde{x}_{ij}: i = 1, \dots, k, j = 1, \dots, l(i)\}$ is a (2ε) -net in \tilde{M} .
- (ii) Let $\{\tilde{x}_1, \dots, \tilde{x}_s\}$ be an ε -net in \tilde{M} . Then the set $P(\{\tilde{x}_1, \dots, \tilde{x}_s\})$ is an ε -net in M (notice that $P(\tilde{x}_i)$ may be equal to $P(\tilde{x}_j)$ even if $i \neq j$).

We omit elementary proofs.

1.3. (p, q) -metrics

Let (M^n, g) be an arbitrary Riemannian manifold with the Levi-Civita connection ∇ . Let E be a vector bundle $\pi: \mathcal{E} \rightarrow M$ equipped with a fibre metric h and a Riemannian connection D . Recall that a fibre metric h it is an inner product h_x on each fibre $E_x = \pi^{-1}(x)$, varying smoothly with $x \in M$. Connection D on E is Riemannian if D is compatible with h , i.e., $Xh(\xi, \eta) = h(D_X\xi, \eta) + h(\xi, D_X\eta)$. In particular, if \mathcal{H} is a subbundle of TM then $\mathcal{H} \rightarrow M$ is a vector bundle, the restriction $g|_{\mathcal{H} \times \mathcal{H}}$ is a fibre metric on \mathcal{H} , and the Levi-Civita ∇ of g induces a Riemannian connection $D = \mathcal{H}\nabla$ on \mathcal{H} . More precisely,

$$D_X\xi = \text{orthogonal projection of } \nabla_X\xi \text{ onto } \mathcal{H},$$

for every vector field X on M and every section ξ of \mathcal{H} .

We define the *connection map* $K = K^D: T\mathcal{E} \rightarrow \mathcal{E}$ related to D as follows: K is a smooth map inducing for every $\zeta \in \mathcal{E}$ a \mathbb{R} -linear map $T_\zeta\mathcal{E} \rightarrow \mathcal{E}_x$, $x = \pi\zeta$ and determined by the condition $K(\xi_*v) = D_v\xi$, $v \in TM$, $\xi \in \Gamma(M, \mathcal{E})$. Notice that by the definition follows that $K|_{T_\zeta\mathcal{E}}$ is the canonical isomorphism $T_\zeta\mathcal{E} \rightarrow \mathcal{E}_x$. For more details on the connection map we refer to [10] and [7].

Following by [2], one can equip $T\mathcal{E}$ with the (p, q) -metric $\mathbf{h}_{p,q}$, $p, q \in \mathbb{R}$, $q \geq 0$, that is, a Riemannian metric defined by:

$$\mathbf{h}_{p,q}(A, B) = g(\pi_*A, \pi_*B) + \frac{1}{(1 + |\zeta|^2)^p} (h(KA, KB) + qh(KA, \zeta)h(KB, \zeta)),$$

where $A, B \in T_\zeta\mathcal{E}$, and $|\zeta|^2 = h(\zeta, \zeta)$.

Suppose $E = (TM, g, \nabla)$. If $p = q = 0$ then $\mathbf{h}_{p,q}$ coincide with the Sasaki metric [7]. If $p = q = 1$ then $\mathbf{h}_{p,q}$ becomes the Cheeger–Gromoll metric [6].

The projection $\pi : (\mathcal{E}, \mathbf{h}_{p,q}) \rightarrow (M, g)$ is a Riemannian submersion such that $\mathcal{H}^\pi = \ker K^D$ and $\mathcal{V}^\pi = \ker \pi_*$. Consequently, for every $w \in T_x M$ and $\xi \in E_x$ there exists a unique horizontal (resp. vertical) vector $w^h \in \mathcal{H}_\xi$ (resp. $w^v \in \mathcal{V}_\xi$), i.e., $\pi_* w^h = w$ and $K w^h = 0$ (resp. $\pi_* w^v = 0$ and $K w^v = w$).

Let $\pi^1 : \mathcal{E}^1 \rightarrow M$ be the unit bundle induced from $\pi : \mathcal{E} \rightarrow M$, i.e., $\mathcal{E}^1 = \{\xi \in \mathcal{E} : |\xi| = 1\}$ and $\pi^1 = \pi|_{\mathcal{E}^1}$. If $\mathcal{E} = TM$ then $\pi^1 : \mathcal{E}^1 \rightarrow M$ is simply the unit sphere bundle SM . If $Q : M \rightarrow N$ is a Riemannian submersion and $\mathcal{E} = \mathcal{H}^Q$ then $\pi^1 : \mathcal{E}^1 \rightarrow M$ is called a unit horizontal bundle.

We define a (p, q) -metric on \mathcal{E}^1 by the restriction of $\mathbf{h}_{p,q}$ to $T(\mathcal{E}^1)$. We denote it also by $\mathbf{h}_{p,q}$. Notice that $\pi^1 = \pi|_{\mathcal{E}^1} : (\mathcal{E}^1, \mathbf{h}_{p,q}) \rightarrow (M, g)$ is a Riemannian submersion such that, for any $\xi \in \mathcal{E}^1$, $\mathcal{H}_\xi^{\pi^1} = \mathcal{H}_\xi^\pi$ and $\mathcal{V}_\xi^{\pi^1} = \{A \in \mathcal{V}_\xi^\pi : h(K^D A, \xi) = 0\}$ (cf. [10, §2, Lemma 1]). Observe that the shape of $\mathcal{V}_\xi^{\pi^1}$ together with the equality $|\xi| = 1$ for $\xi \in \mathcal{E}^1$ cause that $\mathbf{h}_{p,q}$ on \mathcal{E}^1 has the form

$$\mathbf{h}_{p,q}(A, B) = g(\pi_* A, \pi_* B) + 2^{-p} h(K^D A, K^D B),$$

and, in fact, $\mathbf{h}_{p,q}$ no longer depends on q .

Since $\mathbf{h}_{p,q}$ depends on $E = (\pi : \mathcal{E} \rightarrow M, h, D)$ it is convenient to identify E with its total space \mathcal{E} and write $(E, \mathbf{h}_{p,q})$ instead of $(\mathcal{E}, \mathbf{h}_{p,q})$. Moreover, the corresponding unit horizontal bundle with the induced (p, q) -metric is denoted by $(E^1, \mathbf{h}_{p,q})$.

2. Results

2.1. Submersion theorem

Let $P : (\tilde{M}, \tilde{g}) \rightarrow (M, g)$ be a Riemannian submersion, $b = \dim M$, $a + b = \dim \tilde{M}$ and $T\tilde{M} = \mathcal{H}^P \oplus \mathcal{V}^P$ be the corresponding orthogonal splitting of $T\tilde{M}$. Let $\tilde{\nabla}$ and ∇ denote the Levi-Civita connections of \tilde{g} and g , respectively.

Let \tilde{E} denote the horizontal bundle $\mathcal{H}^P \rightarrow \tilde{M}$ equipped with the fibre metric $\tilde{g}|_{\mathcal{H}^P \times \mathcal{H}^P}$ and the Riemannian connection $\tilde{D} = \mathcal{H}\tilde{\nabla}$.

Let $\tilde{\tau}$ be the parallel transport in \tilde{E} , $K^{\tilde{D}}$ and K^∇ be the connection maps corresponding to \tilde{D} and ∇ . Moreover, let $\tilde{\pi}^1$ and π^1 denote the restrictions of the natural projections $\tilde{\pi} : \tilde{E} \rightarrow \tilde{M}$ and $\pi : TM \rightarrow M$ to \tilde{E}^1 and SM , respectively.

For $w \in T_x M$ and $\tilde{x} \in P^{-1}(x)$, let $\tilde{w} = \tilde{w}_{\tilde{x}} \in \mathcal{H}_{\tilde{x}}^P$ be the unique horizontal vector such that $P_* \tilde{w} = w$. Similarly, for any $W \in T_{\tilde{x}} \tilde{M}$ and any $\xi \in \tilde{E}_{\tilde{x}}^1$ let $W^h \in \mathcal{H}_\xi^{\tilde{\pi}^1}$ be the unique horizontal vector such that $\tilde{\pi}_* (W^h) = W$.

If $\tilde{x} \in \tilde{M}$, $P(\tilde{x}) = x$ and $\xi \in \tilde{E}_{\tilde{x}}^1$ then the horizontal lift $\tilde{w}_\xi^h \in \mathcal{H}_\xi^{\tilde{\pi}^1}$ of $w \in T_x M$ to the point ξ may be constructed as follows: Let γ be a curve in M such that $\gamma(0) = x$ and $\dot{\gamma}(0) = w$. Next, let $\tilde{\gamma} = \tilde{\gamma}_{\tilde{x}}$ be the horizontal lift of γ such that $\tilde{\gamma}(0) = \tilde{x}$. Let $\tilde{\gamma}^h(t) = \tilde{\gamma}_t^h(t) = \tilde{\tau}_t^{\tilde{\gamma}} \xi$ be the parallel transport of ξ along $\tilde{\gamma}$ from 0 to t . Then $\tilde{w}_\xi^h = \dot{\tilde{\gamma}}^h(0)$.

Suppose that $p, q \in \mathbb{R}$, $q \geq 0$ are fixed. Let $\mathbf{h} = \mathbf{h}_{p,q}$ and $\tilde{\mathbf{h}} = \tilde{\mathbf{h}}_{p,q}$ denote (p, q) -metrics on SM and \tilde{E}^1 , respectively. We distinguish the following pairwise orthogonal subbundles of $T(\tilde{E}^1)$:

$$\begin{aligned} H'_\xi(\tilde{E}^1) &= T_\xi \tilde{E}_{\tilde{\pi}(\xi)}^1, \\ H''_\xi(\tilde{E}^1) &= \{\tilde{w}_\xi^h : P\tilde{\pi}^1(\xi) = x, w \in T_x M\} = \{\tilde{w}_\xi^h : P\tilde{\pi}(\xi) = x, w \in T_x M\}, \\ V_\xi(\tilde{E}^1) &= \{W^h \in \mathcal{H}_\xi^{\tilde{\pi}^1} : P_* \tilde{\pi}_* W^h = 0\} = \{W^h \in \mathcal{H}_\xi^{\tilde{\pi}^1} : P_* \tilde{\pi}_* W^h = 0\}. \end{aligned}$$

Let $\tilde{P} = P_*$. Since P is a Riemannian submersion we see that

$$\tilde{P} = P_* : \tilde{E}^1 \rightarrow SM.$$

We write $H' = H'(\tilde{E}^1)$, $H'' = H''(\tilde{E}^1)$ and $V = V(\tilde{E}^1)$ for simplicity.

Claim 1. $\tilde{P}_* : H'_\xi \rightarrow T_u(S_x M)$, $u = \tilde{P}\xi$, is an isometry.

Proof. Clearly, $\tilde{P}_*(H'_\xi) \subset T_u S_x M$ and $\dim H'_\xi = \dim T_u(S_x M)$. Thus it suffices to show that \tilde{P}_* preserves the length of vectors.

Let $\tilde{\pi}\xi = \tilde{x}$ and $\pi u = x$. Take $A \in H'_\xi$. Since $A \in \mathcal{V}_\xi^{\tilde{\pi}^1}$ and $\tilde{P}_* A \in \mathcal{V}_u^\pi$,

$$\begin{aligned} |A|^2 &= \frac{1}{2^p} |K^{\tilde{D}} A|^2, \\ |\tilde{P}_* A|^2 &= \frac{1}{2^p} |K^\nabla \tilde{P}_* A|^2. \end{aligned}$$

Since $\tilde{P} : \tilde{E}^1 \rightarrow SM$ is a linear map, $K^\nabla \tilde{P}_* = P_* K^{\tilde{D}}$ on $T_\xi \tilde{E}^1$. Thus

$$|K^\nabla \tilde{P}_* A| = |P_* K^{\tilde{D}} A| = |K^{\tilde{D}} A|.$$

Consequently, $|A| = |\tilde{P}_* A|$. This proves the assertion. \square

Claim 2. $\tilde{P}_* : H''_\xi \rightarrow \mathcal{H}^\pi_u$, $u = P_*\xi$ is an isometry.

Proof. Let $x = \pi u$. By the definition of H'' , $\dim H''_\xi = \dim T_x M$. On the other hand, $\dim \mathcal{H}^\pi_u = \dim T_x M$. To prove the assertion it suffices to show that $\pi_* \tilde{P}_*$ preserves the length of vectors and its image is contained in \mathcal{H}^π_u .

Let $\tilde{w}^h \in H''_\xi$. We can suppose that $\tilde{w}^h = \dot{\gamma}^h(0)$ where γ is a curve in M such that $w = \dot{\gamma}(0)$. Then

$$|\tilde{w}^h| = |\tilde{\pi}_* \tilde{w}^h| = |\tilde{w}| = |P_* \tilde{w}| = |w|.$$

Observe that $\tilde{P} \dot{\gamma}^h$ is a vector field along γ . Indeed, we have

$$\pi \tilde{P} \dot{\gamma}^h = \pi P_* \dot{\gamma}^h = P \tilde{\pi} \dot{\gamma}^h = P \dot{\gamma} = \dot{\gamma}.$$

Next we have

$$|\pi_* \tilde{P}_* \tilde{w}^h| = |(\pi \tilde{P} \dot{\gamma}^h)'(0)| = |\dot{\gamma}'(0)| = |w|.$$

Consequently, we have shown that $|\pi_* \tilde{P}_* \tilde{w}^h| = |\tilde{w}^h|$. Thus $\pi_* \tilde{P}_*$ preserves the length of vectors belonging to H''_ξ .

We have to show that the vertical part of $\tilde{P}_* \tilde{w}^h$ is equal to zero, or equivalently, $K^\nabla(\tilde{P}_* \tilde{w}^h) = 0$. Since $\dot{\gamma}^h(t) \in \mathcal{H}^P$, by Lemma 1.1(i) we conclude that $\nabla_{P_* \dot{\gamma}} P_* \dot{\gamma}^h = P_* \tilde{D}_{\dot{\gamma}} \dot{\gamma}^h$. Since $\tilde{P} \dot{\gamma}^h$ is a vector field along γ we obtain

$$K^\nabla(\tilde{P}_* \tilde{w}^h) = (\nabla_{P_* \dot{\gamma}} P_* \dot{\gamma}^h)(0) = (P_* \tilde{D}_{\dot{\gamma}} \dot{\gamma}^h)(0) = 0.$$

Consequently, $\tilde{P}_* \tilde{w}^h \in \mathcal{H}^\pi_u$. \square

Claim 3. The following conditions are equivalent:

- (i) $\tilde{P}_*(V) = 0$.
- (ii) For every fibre $P^{-1}(x)$ and every curve η in $P^{-1}(x)$, $\tilde{P} \tilde{\tau} \eta = \tilde{P}$.
- (iii) \mathcal{H}^P is integrable.

Proof. (i) \Leftrightarrow (ii). Let $W^h \in V_\xi$, $\xi \in \tilde{E}^1$, $\tilde{\pi} \xi = \tilde{x}$ and let $\tilde{x} \in P^{-1}(x)$. Then $W^h = (t \mapsto \tilde{\tau}_t^\eta \xi)'(0)$, where η is a curve in the fibre $P^{-1}(x)$ such that $\eta(0) = \tilde{x}$ and $\dot{\eta}(0) = \tilde{\pi}_* W^h$. Thus $\tilde{P}_* W^h = (t \mapsto P_* \tilde{\tau}_t^\eta \xi)'(0)$. Consequently, we see that $\tilde{P}_* W^h = 0$ for every $W^h \in V$ iff the curve $t \mapsto P_* \tilde{\tau}_t^\eta \xi$ is constant for every ξ and η . The last condition is equivalent to $P_* \tilde{\tau} \eta = P_*$.

(ii) \Leftrightarrow (iii). By Lemma 1.1(ii) \mathcal{H}^P is integrable iff every basic vector field \tilde{X} is parallel ($\tilde{D} \tilde{X} = 0$). This is equivalent to the condition $\tilde{\tau} \eta \tilde{X} = \tilde{X}$, which is equivalent to (ii). \square

Observe that for any $\xi \in \tilde{E}^1$

$$\dim H'_\xi + \dim H''_\xi + \dim V_\xi = (b-1) + b + a = \dim \tilde{E}^1, \quad (2.1)$$

$$\dim H'_\xi + \dim H''_\xi = (b-1) + b = \dim SM. \quad (2.2)$$

Lemma 2.3. Suppose X and Y are two finite dimensional Euclidean spaces. Suppose Z is a subspace of X and $X = Z \oplus Z^\perp$ is the corresponding orthogonal splitting. Given a linear operator $A : X \rightarrow Y$ such that $A : Z \rightarrow Y$ is an isometry and $A Z^\perp = 0$. If Z' is any linear subspace of X such that $A : Z' \rightarrow Y$ is an isometry then $Z' = Z$.

Proposition 2.1. $\tilde{P} : (\tilde{E}^1, \tilde{\mathbf{h}}) \rightarrow (SM, \mathbf{h})$ is a Riemannian submersion iff \mathcal{H}^P is integrable. Then $\mathcal{H}^{\tilde{P}} = H' \oplus H''$ and $\mathcal{V}^{\tilde{P}} = V$.

Proof. This assertion follows directly by Claims 1–3, (2.1)–(2.2) and Lemma 2.3. \square

2.2. Collapse theorem

Let $P : (\tilde{M}, \tilde{g}) \rightarrow (M, g)$ be a Riemannian submersion and let $f : M \rightarrow (0, +\infty)$ be a smooth function. Put $\tilde{f} = f \circ P$. We modify the Riemannian metric \tilde{g} to \tilde{g}_f putting:

$$\begin{aligned} \tilde{g}_f &= \tilde{g} \quad \text{on } \mathcal{H}^P \times \mathcal{H}^P, \\ \tilde{g}_f &= \tilde{f}^2 \tilde{g} \quad \text{on } \mathcal{V}^P \times T\tilde{M}. \end{aligned}$$

Then $P : (\tilde{M}, \tilde{g}_f) \rightarrow (M, g)$ remains a Riemannian submersion. We call it a *warped submersion*, while the function f is called *warping function* [12]. Denote by M_f the Riemannian manifold (M, g_f) .

Notice that the horizontal distributions of $P : (\tilde{M}, \tilde{g}) \rightarrow (M, g)$ and $P : (\tilde{M}, \tilde{g}_f) \rightarrow (M, g)$ coincide, and are equal to \mathcal{H}^P .

The following *Collapse Theorem* is proved in [12, Thm. 2]. In the whole section ‘lim’ denotes the limit in the GH-topology. For basic facts concerning GH-topology we refer to [9] or [1, §12.4].

Theorem 2.1. Let $P: \tilde{M} \rightarrow M$ be a Riemannian submersion with \tilde{M} compact. Suppose that $(f_n: M \rightarrow (0, +\infty))_{n \in \mathbb{N}}$ is a uniformly bounded sequence of warping functions. Put $\tilde{M}_n = \tilde{M}_{f_n}$. We have $\lim \tilde{M}_n = M$ iff for every $\varepsilon > 0$ there exists a positive integer N such that for every $n > N$ there exists an ε -net $A^{(n)}$ on M such that $f_n|_{A^{(n)}} < \varepsilon$.

Suppose a Riemannian submersion $P: (\tilde{M}, \tilde{g}) \rightarrow (M, g)$ and a warping function f are given. Let $p, q \in \mathbb{R}$ and $q \geq 0$. Let us denote by \tilde{E} the vector bundle $\tilde{\pi}: \mathcal{H}^P \rightarrow \tilde{M}$ with the fibre metric $\tilde{g}|_{\mathcal{H}^P} \times \mathcal{H}^P$. Let \tilde{E}_f denote the vector bundle $\tilde{\pi}: \mathcal{H}^P \rightarrow \tilde{M}$ with the fibre metric $\tilde{g}_f|_{\mathcal{H}^P} \times \mathcal{H}^P$ (see Section 1.3).

Let \tilde{D} and \tilde{D}^f be the connections in \tilde{E} and \tilde{E}_f , respectively. The corresponding connections maps are denoted by K and K^f , respectively.

As in the previous section, let \mathbf{h} and $\tilde{\mathbf{h}}$ denote the (p, q) -metrics on SM and \tilde{E}^1 , respectively. Moreover, let $\tilde{\mathbf{h}}_f$ denote the (p, q) -metric on \tilde{E}_f^1 . We want to find relations between Riemannian manifolds $(\tilde{E}^1, \tilde{\mathbf{h}})$ and $(\tilde{E}_f^1, \tilde{\mathbf{h}}_f)$.

Let $\tilde{P} = P_*$.

Claim 1. $\tilde{P}: (\tilde{E}_f^1, \tilde{\mathbf{h}}_f) \rightarrow (SM, \mathbf{h})$ is a Riemannian submersion iff $\tilde{P}: (\tilde{E}^1, \tilde{\mathbf{h}}) \rightarrow (SM, \mathbf{h})$ is. Then the vertical distributions of these bundles coincide and are equal to $V(\tilde{E}^1)$.

Proof. The first statement of the assertion follows directly from Proposition 2.1 and the fact that the horizontal distribution of a warped submersion and the initial submersion coincide. The second one follows from the fact that the vertical distribution is equal to the kernel of \tilde{P}_* . \square

Claim 2. We have $H'(\tilde{E}_f) = H'(\tilde{E})$ and $H''(\tilde{E}_f) = H''(\tilde{E})$. Consequently, if \mathcal{H}^P is integrable then the horizontal distributions of the Riemannian submersions $\tilde{P}: (\tilde{E}_f^1, \tilde{\mathbf{h}}_f) \rightarrow (SM, \mathbf{h})$ and $\tilde{P}: (\tilde{E}^1, \tilde{\mathbf{h}}) \rightarrow (SM, \mathbf{h})$ coincide.

Proof. The first identity is obvious. The second follows by Lemma 1.1(i). More precisely, suppose γ is a curve in M and $\tilde{\gamma}$ is its horizontal lift. Take any parallel vector field σ along γ . Let $\tilde{\sigma}$ be the unique horizontal vector field along $\tilde{\gamma}$ such that $P_*\tilde{\sigma} = \sigma$. Then Lemma 1.1(i) implies that $\tilde{\sigma}$ is parallel vector field with respect to both connections. It follows that $H''(\tilde{E}_f) = H''(\tilde{E})$. \square

Claim 3. Suppose that \mathcal{H}^P is integrable. Then $\tilde{P}: (\tilde{E}_f^1, \tilde{\mathbf{h}}_f) \rightarrow (SM, \mathbf{h})$ is a warped submersion whose warping function is $\hat{f} = f \circ \pi$, where $\pi: SM \rightarrow M$ is the natural projection.

Proof. Let $\tilde{\xi} \in \tilde{E}_f$, $\tilde{\pi}\tilde{\xi} = \tilde{x}$, $P_*\tilde{\xi} = \xi$ and $P\tilde{x} = x$, and $A, B \in T_{\tilde{\xi}}\tilde{E}_f^1$.

Suppose $A, B \in \mathcal{H}^{\tilde{P}}$. Since H' and H'' are orthogonal we may assume that $A, B \in H'$ or $A, B \in H''$.

In the former case $\tilde{\pi}_*A = \tilde{\pi}_*B = 0$ and $K^fA = KA$, $K^fB = KB$, for A and B are tangent to the fibre at \tilde{x} . Thus, $\tilde{\mathbf{h}}(A, B) = \tilde{\mathbf{h}}_f(A, B)$.

In the latest case, $K^fA = 0$, $K^fB = 0$, and $\tilde{\pi}_*A$ and $\tilde{\pi}_*B$ are members of \mathcal{H}^P . Thus $\tilde{\mathbf{h}}_f(A, B) = \tilde{g}(\pi_*A, \pi_*B) = \tilde{\mathbf{h}}(A, B)$. Consequently, $\tilde{\mathbf{h}}_f = \tilde{\mathbf{h}}$ on $\mathcal{H}^{\tilde{P}} \times \mathcal{H}^{\tilde{P}}$.

If $A, B \in V_{\tilde{\xi}}$ then $\tilde{\pi}_*A$ and $\tilde{\pi}_*B$ are tangent to the fibre $P^{-1}(x)$ at \tilde{x} . Remembering that $\tilde{P}\tilde{\xi} = \xi$, we have

$$\begin{aligned} \tilde{\mathbf{h}}_f(A, B) &= \tilde{g}_f(\tilde{\pi}_*A, \tilde{\pi}_*B) = f^2(x)\tilde{g}(\tilde{\pi}_*A, \tilde{\pi}_*B) \\ &= (f \circ \pi)^2(\xi)\tilde{g}(\tilde{\pi}_*A, \tilde{\pi}_*B) = \hat{f}^2(\xi)\tilde{\mathbf{h}}(A, B). \end{aligned}$$

Thus $\tilde{\mathbf{h}}_f = \hat{f}^2\tilde{\mathbf{h}}$ on $\mathcal{V}^{\tilde{P}} \times \mathcal{V}^{\tilde{P}}$. Since $\mathcal{V}^{\tilde{P}}$ and $\mathcal{H}^{\tilde{P}}$ are mutually orthogonal, $\tilde{\mathbf{h}}_f = \hat{f}^2\tilde{\mathbf{h}}$ on $\mathcal{V}^{\tilde{P}} \times T(\tilde{E}^1)$. \square

After these preparations we can state a collapse theorem for a unit horizontal bundle.

Theorem 2.2. Let $P: (\tilde{M}, \tilde{g}) \rightarrow (M, g)$ be a Riemannian submersion with \tilde{M} compact and integrable horizontal distribution. Suppose that $(f_n: M \rightarrow (0, +\infty))_{n \in \mathbb{N}}$ is an uniformly bounded sequence of warping functions. We equip each unit horizontal bundle $\tilde{E}_{f_n}^1$ and the sphere bundle SM with the (p, q) -metric. Let $\tilde{M}_n = (\tilde{M}, \tilde{g}_{f_n})$ and $\tilde{E}_n^1 = (\tilde{E}_{f_n}^1, \tilde{\mathbf{h}}_{f_n})$. Then $\lim_{n \rightarrow \infty} \tilde{E}_n^1 = SM$ iff $\lim_{n \rightarrow \infty} \tilde{M}_n = M$.

Proof. From Claim 3 it follows that $\tilde{P}: (\tilde{E}_{f_n}^1, \tilde{\mathbf{h}}_{f_n}) \rightarrow (SM, \mathbf{h})$ is a Riemannian submersion. We put $\hat{f}_n = f_n \circ \pi$, where $\pi: SM \rightarrow M$ is the natural projection.

Suppose that $\lim \tilde{M}_n = M$. Take $\varepsilon > 0$. By Theorem 2.1 (\Rightarrow) there exist $N > 0$ such that for every $n > N$ there exist an $(\varepsilon/2)$ -net $A^{(n)} = \{x_1, \dots, x_k\} \subset M$ and $f_n|_{A^{(n)}} < \varepsilon/2$. For each $i = 1, \dots, k$ there exist ξ_{ij} , $j = 1, \dots, l$ such that $\{\xi_{i1}, \dots, \xi_{il}\}$

is an $(\varepsilon/2)$ -net in the fibre S_{x_i} . Put $\hat{A}^{(n)} = \{\xi_{ij} : i = 1, \dots, k; j = 1, \dots, l\}$. By Lemma 1.2(i), $\hat{A}^{(n)}$ is an ε -net in SM . Since $\hat{f}_n(\xi_{ij}) = f_n(x_i) < \varepsilon$, $\hat{f}_n|\hat{A}^{(n)} < \varepsilon$. Consequently, $\lim \tilde{E}_n^1 = SM$ by Theorem 2.1 (\Leftarrow).

Conversely, suppose that $\lim \tilde{E}_n^1 = SM$. Let $\varepsilon > 0$. By Theorem 2.1 (\Rightarrow) there exists $N > 0$ such that for every $n > N$ there exists an ε -net $\hat{A}^{(n)} = \{\zeta_1, \dots, \zeta_m\}$ such that $\hat{f}_n|\hat{A}^{(n)} < \varepsilon$. Then, by Lemma 1.2(ii) $\pi(A^{(n)})$ is an ε -net in M . Moreover, for every $j = 1, \dots, m$, $f(\pi(\zeta_j)) = \hat{f}(\zeta_j) < \varepsilon$. Thus $f|A^{(n)} < \varepsilon$. Consequently, $\lim \tilde{M}_n = M$, by Theorem 2.1 (\Leftarrow). \square

2.3. Fibre products

P.B. Gilkey, J.V. Leahy, and J.H. Park in [5] investigated the relation between the spectrum of Laplacian for total space and base space for Riemannian submersion, in particular for Riemannian submersion with integrable horizontal distribution. The following example comes from that paper. We adopt it to show an application of Theorem 2.2.

Suppose there are given Riemannian submersions $P_i : (\tilde{M}_i, \tilde{g}_i) \rightarrow (M, g)$, $i = 1, \dots, k$. Let $\mathcal{H}^i = \mathcal{H}^{P_i}$ and $\mathcal{V}^i = \mathcal{V}^{P_i}$. Define

$$\tilde{M} = \{\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_k) \in \tilde{M}_1 \times \dots \times \tilde{M}_k : P_1(\tilde{x}_1) = \dots = P_k(\tilde{x}_k)\}.$$

Let $P : \tilde{M} \rightarrow M$ be defined by $P(\tilde{x}) = P_1(\tilde{x}_1)$. Let $\mathcal{V}_{\tilde{x}}^P = \bigoplus_{i=1}^k \mathcal{V}_{\tilde{x}_i}^i$, and $\mathcal{H}_{\tilde{x}}^P = \{\xi = (\xi_1, \dots, \xi_k) \in \bigoplus_{i=1}^k \mathcal{H}_{\tilde{x}_i}^i : P_{1*}(\xi_1) = \dots = P_{k*}(\xi_k)\}$. We define a Riemannian structure \tilde{g} on \tilde{M} by requiring that \mathcal{V}^P and \mathcal{H}^P are orthogonal, $\tilde{g}|_{\mathcal{V}^P \times \mathcal{V}^P}$ is induced from the product metric, and the projection $P_* : \mathcal{H}^P \rightarrow TM$ is an isometry. We call \tilde{g} a *fibre product metric* induced by $P_i : (\tilde{M}_i, \tilde{g}_i) \rightarrow (M, g)$. One can easily check that if all \mathcal{H}^i , $i = 1, \dots, k$, are integrable then \mathcal{H}^P is also integrable.

Let $p, q \in \mathbb{R}$, $q \geq 0$. Suppose that $(f_n : M \rightarrow (0, +\infty))_{n \in \mathbb{N}}$ is a sequence of warping functions. Keeping the notations and conventions from the previous section let $M = (M, g)$, $\tilde{M}_n = (\tilde{M}, \tilde{g}_{f_n})$, $\tilde{M}_{i,n} = (\tilde{M}_i, (\tilde{g}_i)_{f_n})$. Let \mathbf{h} be a (p, q) -metric on SM , $\tilde{\mathbf{h}}_i$ be (p, q) -metrics on unit horizontal bundles \tilde{E}_i^1 , and $\tilde{E}_{i,n}^1 = (\tilde{E}_i^1, (\tilde{\mathbf{h}}_i)_{f_n})$. Moreover, let $\tilde{E}_n^1 = (\tilde{E}^1, \tilde{g}_{f_n})$.

Observe that the warped metric \tilde{g}_{f_n} coincide with the fibre product metric induced by submersions $P_i : \tilde{M}_{i,n} \rightarrow M$.

Suppose that $M, \tilde{M}_1, \dots, \tilde{M}_k$ are compact Riemannian manifolds, and let $P_i : \tilde{M}_i \rightarrow M$ be Riemannian submersions. Let $P : \tilde{M} \rightarrow M$ be a fibre product induced by P_i .

Proposition 2.1. *If all horizontal distributions \mathcal{H}^i , $i = 1, \dots, k$, are integrable, then \mathcal{H}^P is. Moreover, the following conditions are equivalent:*

- (i) $\lim \tilde{M}_{i,n} = M$, for every $i = 1, \dots, k$,
- (ii) $\lim \tilde{M}_n = M$,
- (iii) $\lim \tilde{E}_{i,n}^1 = SM$, for every $i = 1, \dots, k$,
- (iv) $\lim \tilde{E}_n^1 = SM$.

Proof. The equivalences (i) \Leftrightarrow (iii) and (ii) \Leftrightarrow (iv) follow directly from Theorem 2.2. The equivalence (i) \Leftrightarrow (ii) follows easily from Theorem 2.1. \square

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